$$\lim_{\delta=0}\sum_{i=1}^{n}(M_i-m_i)[U(x_i)-U(x_{i-1})]=0.$$

THEOREM V. *In order that a bounded function $f(x)$ shall be integrable from a to b as to a function $u(x)$ of bounded variation, it is necessary and sufficient that the interval (ab) may be divided into partial intervals so that the total variation of $u(x)$ in those in which the oscillation of $f(x)$ is greater than an arbitrarily preassigned positive number ω shall also be as small as one wishes.*

THEOREM VI. *If $u(x)$ is of bounded variation and $f(x)$ is bounded on the interval (ab), then a necessary and sufficient condition for the existence of the integral of $f(x)$ as to $u(x)$ from a to b is that the total variation of $u(x)$ on the set D of discontinuities of $f(x)$ shall be zero (Theorem of Bliss).*

Of the four preceding theorems the only one needing further proof is the last. [For the definition of the total variation of $u(x)$ on a set of points, see Bliss, l. c., p. 633, ll. 12–19.] Let $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ be a sequence of positive numbers decreasing monotonically toward zero, and let $D_1, D_2, D_3, \ldots$ be the closed set of points at which the oscillation of $f(x)$ is $\geqq \epsilon_1, \geqq \epsilon_2, \geqq \epsilon_3, \ldots$ Then the set D of discontinuities of $f(x)$ is the limit of the set D_n when n is indefinitely increased. Now if $f(x)$ is integrable as to $u(x)$ we have seen that the interval (ab) may be divided into partial intervals so that the total variation of $u(x)$ on those in which the oscillation of $f(x)$ is greater than an arbitrarily preassigned positive number shall be as small as one pleases; and this implies that the total variation of $u(x)$ on D_n, and hence on D, is zero. Again, if the total variation of $u(x)$ on D is zero so is it on D_n for every n; and hence $f(x)$ is integrable as to $u(x)$ since it is such that the interval (ab) may be divided into partial intervals so that the sum of those in which the oscillation is greater than an arbitrarily preassigned positive number shall be as small as one pleases.

TRANSFORMATIONS OF CYCLIC SYSTEMS OF CIRCLES

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When two surfaces, S and $\bar{S}$, are applicable, there is a unique conjugate system on S which corresponds to a conjugate system on $\bar{S}$. Denote these conjugate systems, or *nets*, by N and $\overline{N}$ respectively.

The cartesian coördinates x, y, z, of N and $\bar{x}$, $\bar{y}$, $\bar{z}$ of $\bar{N}$ are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u \partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{1}$$

which we call the common *point equation* of N and $\bar{N}$.

Let M and $\bar{M}$ be corresponding points of N and $\bar{N}$. With M as center describe a sphere whose radius is the distance from $\bar{M}$ to the origin. Let Σ_1 and Σ_2 be the sheets of the envelope of the spheres as M moves over N, and let μ_1 and μ_2 be the points of contact with Σ_1 and Σ_2 of the sphere with center at M. The null spheres with centers at μ_1 and μ_2 meet the tangent plane of N in a circle C. These ∞^2 circles from a cyclic system, that is they are orthogonal to ∞^1 surfaces.[1]

If h and l are any pair of solutions of the system

$$\frac{\partial h}{\partial v} = (l-h)\,a, \qquad \frac{\partial l}{\partial u} = (h-l)\,b, \tag{2}$$

the functions x', y', z', defined by the quadratures

$$\frac{\partial x'}{\partial u} = h\frac{\partial x}{\partial u}, \frac{\partial x'}{\partial v} = l\frac{\partial x}{\partial v}; \frac{\partial y'}{\partial u} = h\frac{\partial y}{\partial u}, \frac{\partial y'}{\partial v} = l\frac{\partial y}{\partial v}; \frac{\partial z'}{\partial u} = h\frac{\partial z}{\partial u}, \frac{\partial z'}{\partial v} = l\frac{\partial z}{\partial v}, \tag{3}$$

are the coördinates of a net N' parallel to N, and the functions $\bar{x}'$, $\bar{y}'$, $\bar{z}'$, defined by

$$\frac{\partial \bar{x}'}{\partial u} = h\frac{\partial \bar{x}}{\partial u}, \frac{\partial \bar{x}'}{\partial v} = l\frac{\partial \bar{x}}{\partial v}; \frac{\partial \bar{y}'}{\partial u} = h\frac{\partial \bar{y}}{\partial u}, \frac{\partial \bar{y}'}{\partial v} = l\frac{\partial \bar{y}}{\partial v}; \frac{\partial \bar{z}'}{\partial u} = h\frac{\partial \bar{z}}{\partial u}, \frac{\partial \bar{z}'}{\partial v} = l\frac{\partial \bar{z}}{\partial v}, \tag{4}$$

are the coördinates of a net $\bar{N}'$ parallel to $\bar{N}$. Moreover, the nets N^2 and $\bar{N}'$ are applicable, and consequently the function $\theta' = \Sigma x'^2 - \Sigma \bar{x}''$ is a solution of the point equation of N' and $\bar{N}'$. By the quadratures

$$\frac{\partial\theta}{\partial u} = \frac{1}{h}\frac{\partial\theta'}{\partial u}, \quad \frac{\partial\theta}{\partial v} = \frac{1}{l}\frac{\partial\theta'}{\partial v}$$

we obtain a solution θ of (1).

The functions x_1, y_1, z_1, and $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$, defined by equations of the form

$$x_1 = x - \frac{\theta}{\theta'}x', \qquad \bar{x}_1 = \bar{x} - \frac{\theta}{\theta'}\bar{x}' \tag{5}$$

are the cartesian coördinates of two applicable nets N_1 and $\bar{N}_1$, which are T transforms of N and $\bar{N}_1$ respectively; a net and a T transform are such that the developables of the congruence of lines joining corresponding points of the two nets meet the surfaces on which the nets lie in these nets.[2]

Since N_1 and $\bar{N}_1$ are applicable nets, we can obtain a cyclic system of circles C_1, in the manner described in the second paragraph. Hence each pair of solutions of (2) determines a transformation of the cyclic system of circles C into a cyclic system of circles C_1, such that the nets enveloped by the planes of the circles C and C_1 are in the relation of a transformation T. Moreover, it can be shown that corresponding circles lie on a sphere. We say that two such cyclic systems are in relation T.

Darboux[3] has stated the results of the second paragraph in the following form: If a surface $\bar{S}$ rolls over an applicable surface S, and Q is a point invariably fixed to S, the isotropic generators of the null sphere with center Q meet the plane of contact of S and $\bar{S}$ in points of a circle C which generate the surfaces orthogonal to the cyclic system of circles C. Making use of these ideas, we give the following interpretation of the above transformations of cyclic systems:

If N and $\bar{N}$ are applicable nets, and N_1 and $\bar{N}_1$ are respective T transforms by means of (5), where $\theta' = \Sigma x'^2 - \Sigma \bar{x}'^2$, the cyclic systems in which a point sphere invariably bound to $\bar{N}$ and $\bar{N}_1$ meets the planes of contact, as $\bar{N}$ rolls on N and $\bar{N}_1$ on N_1, are in relation T.

It can be shown that the two surfaces orthogonal to these respective cyclic systems which are generated by the points where an isotropic generator of the null sphere meets the plane of contact are in the relation of a transformation of Ribaucour, that is, these surfaces are the sheets of the envelope of a two parameter family of spheres and the lines of curvature on the two sheets correspond.

[1] Guichard, *Ann. Sci. Ec. norm., Paris*, (Ser. 3), 20, 1903, (202).
[2] Eisenhart, these PROCEEDINGS, 3, 1917, (637).
[3] *Leçons sur la théorie génêrale des surfaces*, vol. 4, 123.